ISI – Bangalore Center – B Math - Physics I – Mid Semester Exam
Date: 8 May 2019. Duration of Exam: 3 hours
Total marks:90

Part A – Do any 2 OUT OF 3 in this part. Each question is 15 marks

Q1.

a.)For a rigid body which has z-axis as the axis of symmetry, determine which of the independent elements of the moment of inertia matrix must be zero. Hence write down a typical moment of inertia matrix for such a rigid body.

b.)Show that the axis of symmetry for a rigid body is always a principal axis.

c.) If the moment of inertia matrix is I in one frame and I' in another frame related to the first one by rotation matrix R how are I, I' related? Justify your answer.

Q2.

A particle of mass m moves along the x axis and is acted upon by the restoring force $-m(n^2+k^2)x$ and the resistance force $-2mk\dot{x}$ where n,k are positive constants. If the particle is released from rest at x = a show that in the subsequent motion

$$x = \frac{a}{n}e^{-kt}(n\cos nt + k\sin nt).$$

Find how far the particle travels before it next comes to rest.

Q3.

The engines of a far away spaceship have failed and the ship is moving in a straight line with speed V far from a planet. The crew has calculated that that present course will miss the planet by a distance p if there is no interaction between the spaceship and the planet. But the planet is known to exert a force on any mass m given by

$$\boldsymbol{F} = -\frac{m\gamma}{r^3}\hat{\boldsymbol{r}}, \quad \gamma = \frac{8p^2V^2}{9}.$$

Show that the spaceship will go around the planet before continuing along the original path.

Show that the distance of closest approach is $p/3$ and the speed of the spaceship as that instant is $3V$.

Hint: use the path equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{f(1/u)}{L^2u^2},$$

$$F = mf(r)\hat{r}.$$

PART B. ANSWER 3 OUT OF 4 Questions – Each Question 20 marks.

Q4

a.) Suppose a system of particles is described by a set of generalized coordinates q_i s which are interacting under mutual interactions described by a potential that depends only on generalized coordinates. Assume that a stable configuration exists and corresponds to q_i=0 for all the coordinates and the potential energy is zero at that point.

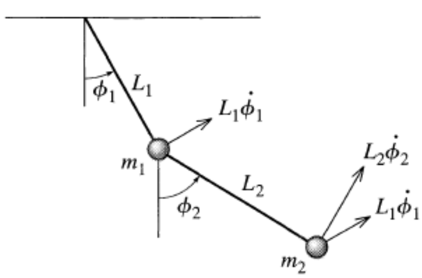
This system for small oscillations can be described by the Lagrangian using usual notation (you need not derive this)

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = T(\dot{\mathbf{q}}) - U(\mathbf{q}),$$

$$U = U(\mathbf{q}) = \tfrac{1}{2}\sum_{j,k} K_{jk}\, q_j\, q_k \quad T = T(\dot{\mathbf{q}}) = \tfrac{1}{2}\sum_{j,k} M_{jk}\, \dot{q}_j\, \dot{q}_k.$$

Explain why the matrix ***K*** and ***M*** are positive definite.

b.) Apply the result of part a.) and determine the matrix ***K*** and ***M*** for a double pendulum shown in figure.



You can use the following expressions for the kinetic and potential energies.

$$T = \tfrac{1}{2}(m_1 + m_2)L_1^2\dot{\phi}_1^2 + m_2L_1L_2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1 - \phi_2) + \tfrac{1}{2}m_2L_2^2\dot{\phi}_2^2.$$

$$U(\phi_1, \phi_2) = (m_1 + m_2)gL_1(1 - \cos\phi_1) + m_2gL_2(1 - \cos\phi_2)$$

c.) Show the normal frequencies are given by $\frac{3}{4}\omega_0^2, \frac{3}{2}\omega_0^2$ where $\omega_0^2 = L/g$ and determine the normal modes when $m_1 = 8m, m_2 = m, L_1 = L_2 = L$.

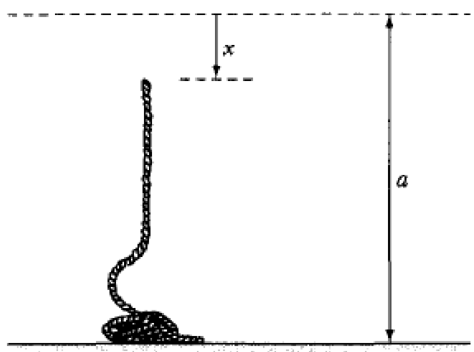
d.) Find the actual motion $[\phi_1(t), \phi_2(t)]$ if the pendulum is dropped from rest with $[\phi_1(t=0) = 0, \phi_2(t=0) = \alpha]$. Is the motion periodic?

Q5.

a.) Two particles interact with each other via a common potential $V(|\vec{r}_1 - \vec{r}_2|)$ .The force on particle 1 and particle 2 are given by $-\vec{\nabla}_1 V$ and $-\vec{\nabla}_2 V$ respectively. There are no external forces. . Either prove or give an counterexample of the following statement: The trajectories of the two particles are confined to a plane.

b.) Express the kinetic energy of the above system as a function of their masses, the velocity of the center of mass, and the relative velocity of the particles. Hence show that in an elastic collision of two particles, the magnitude of the relative velocity remains the same before and after the collision.

c.) Consider a rope of mass per unit length ρ and length a suspended just above a table as shown in the picture. If the rope is released from rest at the top, find the force on the table when a length x of the rope has dropped to the table.



Hint, note that at every instant there is a bit of the rope that comes to an abrupt stop on the table due to the impulse force exerted by the table. The table will experience an equal and opposite force.

d.) Is this an example of an elastic collision or an inelastic collision?

Q6.

a.) Starting with the equation

$$\mathbf{v}^{\text{in}} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}.$$

relating the velocity $\mathbf{v}^{\text{in}}$ in a fixed frame with the velocity **v** as observed in a rotating frame show that the effective force in rotating frame is given by the following equation

$$m\mathbf{a} = \mathbf{F} - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Here **a** and **v** are accelerations and velocity in the rotating frame and $\vec{\omega}$ is the angular velocity of the rotating frame.

b.) Using the above result show that the small angle deviation ε of a plumb line from the true vertical (that is the line towards the centre of the earth) at a point on the earth surface at the latitude lambda is given by

$$\varepsilon = \frac{R\omega^2 \sin\lambda \cos\lambda}{g_0 - R\omega^2 \cos^2\lambda}$$

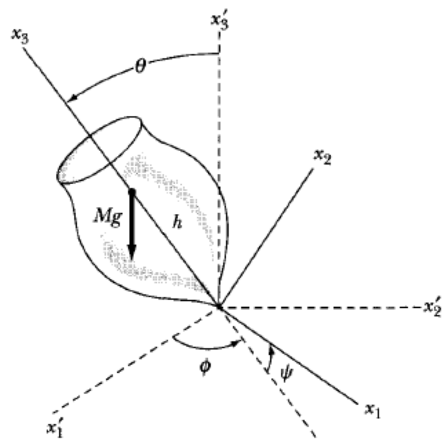
where **R** is the radius of the earth taken to be spherical and $\mathbf{g_0}$ is the acceleration due to gravity. Assume that the angular velocity of the Earth is constant. Show maximum deviation happens when

$$\cos 2\overline{\lambda} = -\alpha/(2-\alpha).$$

Here $\alpha = R\omega^2 / g_0$, and is a very small number for the earth. At which latitude will you expect the deviation to be maximum on the earth?

Q7.

The Lagrangian for a heavy symmetric top is given in usual notation by



$$L = \frac{1}{2} I_1(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgh\cos\theta$$

Find the expressions of the conserved momenta p_ϕ, p_ψ associated with ϕ, ψ.

Using the fact without derivation that energy conservation leads to the equation for theta is given by

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta)$$

,

$$V(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + Mgh\cos\theta$$

b.)Show that the motion in the variable theta is bounded between two values.

c.)Show that there exists a solution with a constant $\theta = \theta_0$ provided.

$$p_\psi^2 \geq 4MghI_1 \cos\theta_0$$

.

d.)Physically what does this condition mean? Describe the motion when this condition is satisfied.

e.) Notice that the condition in c.) is always satisfied if there is no gravity. What does it say about the motion of a free top?